Trigonometric Equations and Identities

January 2014

Question 1 — 2 marks

Find the coterminal angle to $\frac{27\pi}{5}$ over the interval $[-360°, 0°)$.

Solution

Method 1

$$\frac{27\pi}{5}\left(\frac{180^\circ}{\pi}\right) = 27\left(36^\circ\right)$$

$$= 972^\circ$$

1 mark for conversion to degrees

$$972^\circ - \left(360^\circ\right)(3) = -108^\circ$$

½ mark for coterminal angle
½ mark for correct domain

2 marks

Method 2

$-\frac{3\pi}{5}$ is a coterminal angle to $\frac{27\pi}{5}$.

½ mark for coterminal angle
½ mark for correct domain

$$-\frac{3\pi}{5} \cdot \frac{180^\circ}{\pi} = -108^\circ$$

1 mark for conversion to degrees

2 marks

Question 2 (calculator) 3 marks

Solve the following equation over the interval $0 \leq \theta < 2\pi$.

$$(\tan\theta - 3)(\tan\theta + 1) = 0$$

Solution

$$(\tan\theta - 3)(\tan\theta + 1) = 0$$

$\tan\theta = 3$ $\quad$ $\tan\theta = -1$ — 1 mark (½ mark for each branch)

$\theta_r = 1.249\ 046$ $\quad$ $\theta_r = \frac{\pi}{4}$

$\theta = 1.249\ 046$ $\quad$ $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$ — 2 marks (½ mark for each value of θ)

$\theta = 4.390\ 639$ $\quad$ **or**

$\theta = 1.249, 4.391$ $\quad$ $\theta = 2.356\ 194$

$\theta = 5.497\ 787$

3 marks

$$\therefore \theta = 1.249, \frac{3\pi}{4}, 4.391, \frac{7\pi}{4}$$

or

$$\theta = 1.249, 2.356, 4.391, 5.498$$

Question 7 4 marks

Solve the following equation over the interval $[0, 2\pi]$.

$$2\cos 2\theta - 1 = 0$$

Solution

Method 1

$$2\cos 2\theta - 1 = 0$$

$$2\left(2\cos^2\theta - 1\right) - 1 = 0$$ 1 mark for identity

$$4\cos^2\theta - 3 = 0$$

$$\cos^2\theta = \frac{3}{4}$$

$$\cos\theta = \pm\frac{\sqrt{3}}{2}$$ 1 mark for solving for $\cos\theta$

$$\theta_r = \frac{\pi}{6}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$ 2 marks for solutions (½ mark for each consistent solution)

4 marks

Question 12 a) 2 marks b) 2 marks

Solution

Method 1

a)

LHS	RHS
$\frac{1}{\cos^2\theta}+\frac{2\cos^2\theta}{\cos^2\theta}$	$\tan^2\theta+3$
$\sec^2\theta+2$	
$\tan^2\theta+1+2$	
$\tan^2\theta+3$	

$\therefore$ LHS = RHS

1 mark for appropriate algebraic strategy
1 mark for appropriate identity substitutions

2 marks

b) $\cos^2\theta=0$

$\cos\theta=0$

$\theta=\frac{\pi}{2},\frac{3\pi}{2},\ldots$

$\therefore$ the non-permissible values of θ are $\frac{\pi}{2}+k\pi,\ k\in I$

or

$90^\circ+180k,\ k\in I.$

½ mark for $\cos^2\theta=0$

½ mark for any non-permissible value of θ

1 mark for all non-permissible values of θ

2 marks

Solution

Method 2

a)

LHS	RHS
$\frac{1+2\cos^2\theta}{\cos^2\theta}$	$\tan^2\theta+3$
	$\sec^2\theta-1+3$
	$\sec^2\theta+2$
	$\frac{1}{\cos^2\theta}+2$
	$\frac{1}{\cos^2\theta}+\frac{2\cos^2\theta}{\cos^2\theta}$
	$\frac{1+2\cos^2\theta}{\cos^2\theta}$

$\therefore$ LHS = RHS

1 mark for appropriate identity substitutions
1 mark for appropriate algebraic strategy

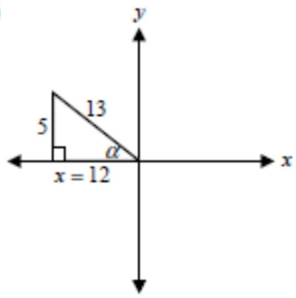
2 marks

Question 14 a) 3 marks b) 1 mark

Given that $\sin\alpha = \frac{5}{13}$, where α is in Quadrant II, and $\cos\beta = \frac{2}{5}$, where β is in Quadrant IV, find the exact value of:

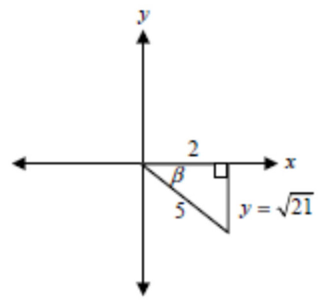
Solution

a)



$$\begin{aligned} x^2 + y^2 &= r^2 \\ x^2 + 25 &= 169 \\ x^2 &= 144 \\ x &= \pm 12 \\ x &= -12 \end{aligned}$$

$$\begin{aligned} x^2 + y^2 &= r^2 \\ 4 + y^2 &= 25 \\ y^2 &= 21 \\ y &= \pm\sqrt{21} \\ y &= -\sqrt{21} \end{aligned}$$



½ mark for value of x
½ mark for value of y

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \left(-\frac{12}{13}\right)\left(\frac{2}{5}\right) - \left(\frac{5}{13}\right)\left(-\frac{\sqrt{21}}{5}\right) \\ &= -\frac{24}{65} + \frac{5\sqrt{21}}{65} \\ &= \frac{5\sqrt{21} - 24}{65} \end{aligned}$$

½ mark for $\cos\alpha$
½ mark for $\sin\beta$

1 mark for substitution into correct identity

3 marks

b)

$$\begin{aligned} \sin 2\alpha &= 2\sin\alpha\cos\alpha \\ &= 2\left(\frac{5}{13}\right)\left(-\frac{12}{13}\right) \\ &= -\frac{120}{169} \end{aligned}$$

1 mark for substitution into correct identity

1 mark

June 2013

Question 2 (Calculator) 4 marks

Solve the equation $csc^2\theta + 3\csc\theta - 4 = 0$ over the interval $[0, 2\pi]$.
Express your answers as exact values or correct to 3 decimal places.

Solution

Method 1

$$\csc^2\theta + 3\csc\theta - 4 = 0$$
$$(\csc\theta - 1)(\csc\theta + 4) = 0$$

$\csc\theta = 1$ $\quad$ $\csc\theta = -4$ $\quad$ 1 mark for solving for $\csc\theta$

$\sin\theta = 1$ $\quad$ $\sin\theta = -\frac{1}{4}$ $\quad$ 1 mark for reciprocal of $\csc\theta$

$\theta = \frac{\pi}{2}$ $\quad$ $\theta_r = 0.252\ 680$

or

$\theta = 1.570\ 796$ $\quad$ $\theta = 3.394\ 273,\ 6.030\ 505$

$\theta = \frac{\pi}{2}, 3.394, 6.031$ $\quad$ 2 marks (1 mark for consistent solutions of each trigonometric equation)

or

$\theta = 1.571, 3.394, 6.031$

4 marks

Question 7 4 marks

Solve the following equation algebraically where $180° \leq \theta \leq 360°$.

$$2sin^2\theta + 5\cos\theta + 1 = 0$$

Solution

$2\left(1-\cos^2\theta\right)+5\cos\theta+1=0$		1 mark for identity
$2-2\cos^2\theta+5\cos\theta+1=0$		
$2\cos^2\theta-5\cos\theta-3=0$		
$\left(2\cos\theta+1\right)\left(\cos\theta-3\right)=0$		
$\cos\theta=-\frac{1}{2}$	$\cos\theta=3$	1 mark for solving for $\cos\theta$
$\theta_r=60°$	$\therefore$ no solution	1 mark for indicating no solution
$\theta=240°$		1 mark for solving for θ

4 marks

Question 15 3 marks

Prove the identity below for all permissible values of x.

$$\frac{sin^2x}{\sec x+1} = \cos x - cos^2x$$

Solution

Method 1

$$\text{LHS} = \frac{1-\cos^2 x}{\frac{1}{\cos x}+1}$$

$$= \frac{1-\cos^2 x}{\frac{1+\cos x}{\cos x}}$$

$$= \left(1-\cos^2 x\right)\left(\frac{\cos x}{1+\cos x}\right)$$

$$= (1-\cos x)(1+\cos x)\left(\frac{\cos x}{1+\cos x}\right)$$

$$= (1-\cos x)(\cos x)$$

$$= \cos x - \cos^2 x$$

$$= \text{RHS}$$

1 mark for correct substitution of identities
1 mark for algebraic strategies
1 mark for logical process to prove the identity

3 marks

Question 39 2 marks

On the interval $0 \leq \theta < 2\pi$, identify the non-permissible values of θ for the trigonometric identity:

$$\tan\theta = \frac{1}{\cot\theta}$$

Solution

$$\frac{\sin\theta}{\cos\theta} = \frac{1}{\frac{\cos\theta}{\sin\theta}}$$

$\therefore$ the above identity is non-permissible when $\cos\theta = 0$ or $\sin\theta = 0$.

1 mark for identifying non-permissible values
(½ mark for $\cos\theta = 0$, ½ mark for $\sin\theta = 0$)

$\cos\theta \neq 0$

$\theta \neq \frac{\pi}{2}, \frac{3\pi}{2}$

$\sin\theta \neq 0$

$\theta \neq 0, \pi$

1 mark for solving for θ (½ mark for each solution set)

2 marks

$\theta \neq 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$