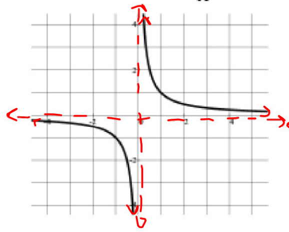
Lesson 1 Sketching Graphs of Rational Functions

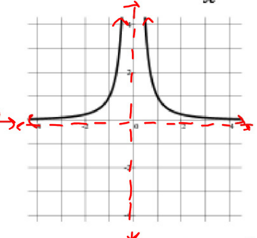
Basic Shapes of Rational Functions

Sketch: $y = \frac{1}{x}$



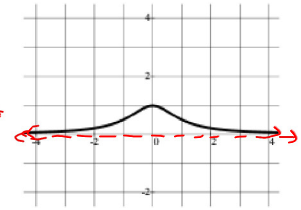
hyperbola

Sketch: $y = \frac{1}{x^2}$



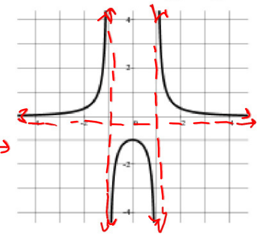
inverted funnel

Sketch: $y = \frac{1}{x^2+1}$



speed bump

Sketch: $y = \frac{1}{x^2-1}$



H-shape

A rational function has the form $y = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$.

Characteristics of Rational Functions

1. Non-permissible values of *x*

- **Vertical Asymptote**: a vertical line that the graph approaches but never touches
- **Hole**: a point of discontinuity

Horizontal Asymptotes (H.A.)

For $y = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ have no common factors, the following happens:

m is degree of p(x)
n is degree of q(x)

- If the degree of $p(x)$ is less than the degree of $q(x)$, the H.A. is at $y = 0$ $m < n$
- If the degree of $p(x)$ is equal to the degree of $q(x)$, the H.A. is at $y = \frac{a}{b}$, where "a" is the leading coefficient of $p(x)$ and "b" is the leading coefficient of $q(x)$ $m = n$
- If the degree of $p(x)$ is greater than the degree of $q(x)$ then the graph won't have a H.A. $m > n$

Steps to Sketch a Rational Function

1. Plot the y-intercept.
   - Set $x = 0$, solve for y
2. Plot x-intercept(s).
   - Set $y = 0$ and solve.
     (Shortcut: Set the numerator equal to 0 and solve.)
3. Sketch the vertical asymptotes.
   - Set the denominator equal to 0 and solve.
4. Sketch the horizontal asymptote.
   - Compare degrees of the numerator and denominator.
5. Determine the behavior of function near asymptotes.
   - Plot a point (or points) in each section of the graph using a table of values.
6. Use smooth curves to complete the graph, approaching the asymptotes as x approaches $\pm\infty$.

$y = \frac{1}{x+1}$

$0 = \frac{1}{\cancel{x+1}}$

$0 \neq 1$

$y = \frac{x+3}{x+1}$

$0 = \frac{x+3}{\cancel{x+1}}$

$0 = x+3$

$x = -3$

Ex. 1) Sketch the graph of $y = \frac{2x}{x+4}$. State the domain and range.

y-int

$y = \frac{2(0)}{0+4}$

$= 0$

x-int

num $2x = 0$

$x = 0$

V.A.

denom $x + 4 = 0$

$x = -4$

H.A.

$m = 1$

$n = 1$

$m = n$

$y = \frac{a}{b}$

$y = \frac{2}{1}$

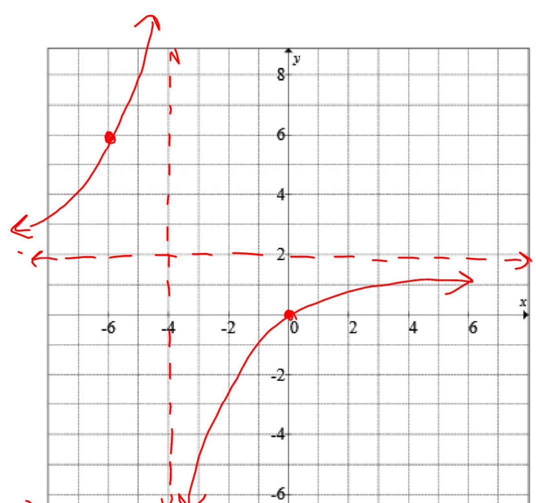
$y = 2$

$x = -6$

$y = \frac{2(-6)}{-6+4}$

$y = 6$

pt $(-6, 6)$



D: $x \neq -4$

R: $y \neq 2$

Ex. 2) Sketch the graph of $y = \frac{2x^2}{x^2-4}$. ← U-shape

State the domain and range.

y-int

$y = \frac{2(0)^2}{0^2-4}$

$= 0$

x-int

$2x^2 = 0$

$x^2 = 0$

$x = 0$

V.A.

$x^2 - 4 = 0$

$x^2 = 4$

$x = \pm 2$

H.A.

$m = 2$

$n = 2$

$\therefore y = \frac{2}{1}$

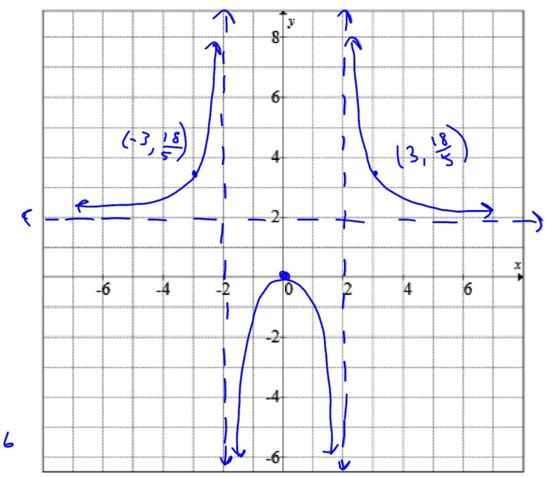
$y = 2$

$x = \pm 3$

$y = \frac{2(\pm 3)^2}{(\pm 3)^2 - 4}$

$y = \frac{18}{5}$

$y = 3\frac{3}{5} \approx 3.6$



D: $x \neq \pm 2$

* R: $(-\infty, 0] \cup (2, \infty)$

Ex. 3) Sketch the graph of $g(x) = \frac{4}{x^2+4}$. ← speed bump

State the domain and range.

y-int

$g(0) = \frac{4}{0^2+4}$

$= 1$

x-int

$4 \neq 0$

$\varnothing$

$\therefore$ no x-int

V.A.

$x^2 + 4 = 0$

$x^2 = -4$

$\varnothing$

no V.A.

H.A.

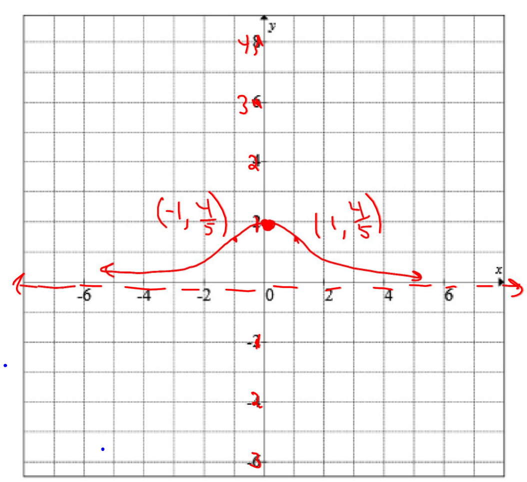
$m < n$

$y = 0$

$x = \pm 1$

$g(\pm 1) = \frac{4}{(\pm 1)^2 + 4}$

$= \frac{4}{5}$



D: $x \in \mathbb{R}$

R: $0 < y \leq 1$

or

D: $(-\infty, \infty)$

R: $(0, 1]$

Ex. 2) Sketch the graph of $y = \frac{1}{2x^2}$. ← inverted funnel
State the domain and range.

y-int
$y = \frac{1}{2(0)^2}$
$\varnothing$
no y-int

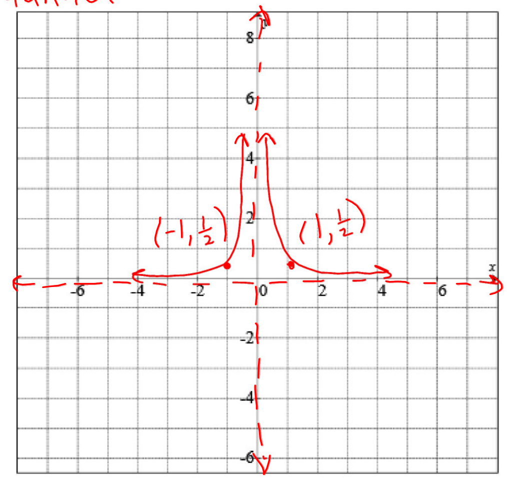
x-int
$1 \neq 0$
no x-int

V.A.
$2x^2 = 0$
$x^2 = 0$
$x = 0$

H.A.
$m < n$
$y = 0$

$x = \pm 1$
$y = \frac{1}{2(\pm 1)^2}$
$= \frac{1}{2}$

D: $x \neq 0$
R: $y > 0$



Sketching Using Transformations

$$y = \frac{a}{x-h} + k$$

Example 1
Describe the transformations which would be used to sketch the graph of
$y = \frac{6}{x-2} - 3$.

$y = \frac{1}{x}$ ← basic hyperbola

vertical stretch by a factor of 6
horizontal translation to the right 2 units
vertical translation down 3