Related Rates — They Rate, Relatively

Say you're filling up your swimming pool and you know how fast water is coming out of your hose, and you want to calculate how fast the water level in the pool is rising. You know one rate (how fast the water is being poured in), and you want to determine another rate (how fast the water level is rising). These rates are called *related rates* because one depends on the other — the faster the water is poured in, the faster the water level will rise. In a typical related rates problem, the rate or rates you're given are unchanging, but the rate you have to figure out is changing with time. You have to determine this rate at one particular point in time.

Solving these problems can be tricky at first, but with practice you get the hang of it. The strategies and tips I discuss are a big help. Now for three examples.

Blowing up a balloon

You're blowing up a balloon at a rate of 300 *cubic inches per minute.* When the balloon's radius is 3 inches, how fast is the radius increasing?

1. **Draw a diagram, labeling the diagram with any *unchanging* measurements (there aren't any in this unusually simple problem) and making sure to assign a variable to anything in the problem that's *changing* (unless it's irrelevant to the problem.) See Figure 12-5.**

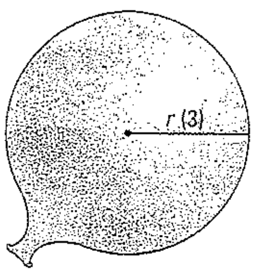

Figure 12-5: Blowing up a balloon — time to party.

Notice that the radius in Figure 12-5 is labeled with the variable r. The radius needs a variable because as the balloon is being blown up, the radius is *changing*. I put the 3 in parentheses to emphasize that the number 3 is *not* an unchanging measurement. The problem asks you to determine something *when* the radius is 3 inches, but remember, the radius is constantly changing.

TIP: In related rates problems, it's important to distinguish between what is changing and what is *not* changing.

The volume of the balloon is also changing, so you need a variable for volume, V. You could put a V on your diagram to indicate the changing volume, but there's really no easy way to mark part of the balloon with a V like you can show the radius with an r.

2. **List all given rates and the rate you're asked to determine as derivatives with respect to time.**

You're pumping up the balloon at 300 *cubic inches per minute.* That's a rate — it's a change in volume (cubic inches) per change in time (minutes). So,

$$\frac{dV}{dt} = 300 \textit{ cubic inches per minute}$$

You have to figure out how fast the radius is changing, so

$$\frac{dr}{dt} = ?$$

3. **Write down the formula that connects the variables in the problem, *V* and *r*.**

   Here's the formula for the volume of a sphere:

   $$V = \frac{4}{3}\pi r^3$$

4. **Differentiate your formula with respect to time, *t*.**

   This works like implicit differentiation because you're differentiating with respect to *t*, but the formula is based on something else, namely *r*.

   $$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt}$$

   $$= 4\pi r^2 \frac{dr}{dt}$$

   You get a $\frac{dr}{dt}$ just like you get a y' or a $\frac{dy}{dx}$ with implicit differentiation.

5. **Substitute known values for the rate and variables in the equation from Step 4, and then solve for the thing you're asked to determine.**

   It's given that $\frac{dV}{dt} = 300$, and you're asked to figure out $\frac{dr}{dt}$ when $r = 3$, so plug in these numbers and solve for $\frac{dr}{dt}$.

   WARNING!

   Be sure to differentiate (Step 4) *before* you plug the given information into the unknowns (Step 5).

   $$300 = 4\pi \cdot 3^2 \frac{dr}{dt}$$

   $$300 = 36\pi \frac{dr}{dt}$$

   $$\frac{300}{36\pi} = \frac{dr}{dt}$$

   $$\frac{dr}{dt} \approx 2.65 \textit{ inches per minute}$$

So the radius is increasing at a rate of about 2.65 *inches per minute* when the radius measures 3 inches. Think of all the balloons you've blown up since your childhood. Now you finally have the answer to the question that's been bugging you all these years.

By the way, if you plug 5 into *r* instead of 3, you get an answer of about 0.95 *inches per minute*. This should agree with your balloon-blowing-up experience — the bigger the balloon gets, the slower it grows. It's a good idea to check things like this every so often to see that the math agrees with your common sense.

Filling up a trough

Here's a garden-variety related rates problem. A trough is being filled up with swill. It's 10 feet long, and its cross-section is an isosceles triangle with a base of 2 feet and a height of 2 feet 6 inches (with the vertex at the bottom, of course). Swill's being poured in at a rate of 5 *cubic feet per minute*. When the depth of the swill is 1 foot 3 inches, how fast is the swill level rising?

1. **Draw a diagram, labeling the diagram with any *unchanging* measurements and assigning variables to any *changing* things. See Figure 12-6.**

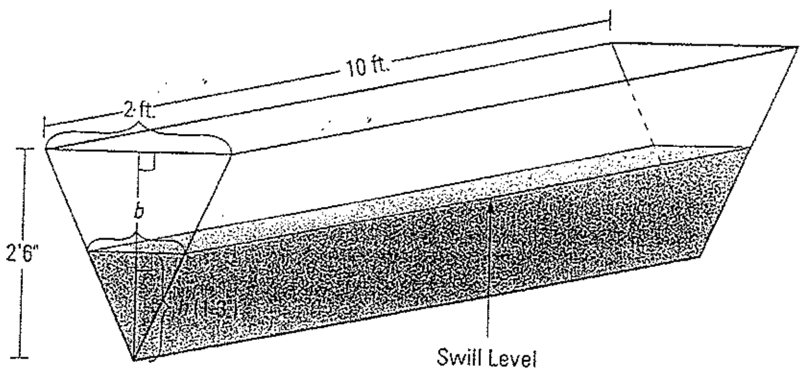

Figure 12-6: Filling a trough with swill — lunch time.

(Note: The perspective is not quite right, so you can see the exact shape of the triangle.)

Note that Figure 12-6 shows the *unchanging* dimensions of the trough, 2 feet, 2 feet 6 inches, and 10 feet, and that these dimensions do *not* have variable names like *l* for length or *h* for height. And note that the *changing* things — the height (or depth) of the swill and the width of the surface of the swill (which gets wider as the swill gets deeper) — have variable names, *h* for height and *b* for base (I call it the *base* instead of the *width* because it's the base of the upside-down triangle shape made by the swill). The volume of the swill is also changing, so you can call that *V*, of course.

2. **List all given rates and the rate you're asked to figure out as derivatives with respect to *time*.**

$$\frac{dV}{dt} = 5 \text{ cubic feet per minute}$$

$$\frac{dh}{dt} = ?$$

3.a. **Write down the formula that connects the variables in the problem: *V*, *h*, and *b*.**

I'm *absolutely positive* that you remember the formula for the volume of a right prism (the shape of the swill in the trough):

$$V = (\text{area of base})(\text{height})$$

Note that this "base" is the base of the prism (the whole triangle at the end of the trough), not the base of the triangle which is labeled *b* in Figure 12-6. Also, this "height" is the height of the prism (the length of the trough), not the height labeled *h* in Figure 12-6. Sorry about the confusion. Deal with it.

The area of the triangular base equals $\frac{1}{2}bh$ and the "height" of the prism is 10 feet, so the formula becomes

$$V = \frac{1}{2}bh \cdot 10$$

$$V = 5bh$$

Now, unlike the formula in the balloon example, this formula contains a variable, *b*, that you don't see in your list of derivatives in Step 2. So Step 3 has a second part — getting rid of this extra variable.

3.b. **Find an equation that relates the unwanted variable, b, to some other variable in the problem so you can make a substitution that leaves you with only V and h.**

The triangular face of the swill in the trough is *similar* to the triangular face of the trough itself, so the base and height of these triangles are proportional. (Recall from geometry that *similar triangles* are triangles of the same shape; their sides are proportional.) Thus,

$$\frac{b}{2} = \frac{h}{2.5}$$

$$2.5b = 2h \quad \text{(cross multiplication)}$$

$$b = \frac{2h}{2.5}$$

$$b = 0.8h$$

Similar triangles come up a lot in related rates problems. Look for them whenever the problem involves a triangle, a triangular prism, or a cone shape.

Now substitute $0.8h$ for b in your formula from Step 3.a.

$$V = 5bh$$

$$V = 5 \cdot 0.8h \cdot h$$

$$V = 4h^2$$

4. **Differentiate this equation with respect to t.**

$$\frac{dV}{dt} = 8h\frac{dh}{dt}$$

5. **Substitute known values for the rate and variable in the equation from Step 4 and then solve.**

You know that $\frac{dV}{dt} = 5$ *cubic feet per minute,* and you want to determine $\frac{dh}{dt}$ when h equals 1 foot 3 inches, or 1.25 feet, so plug in 5 and 1.25 and solve for $\frac{dh}{dt}$.

$$\frac{dV}{dt} = 8h\frac{dh}{dt}$$

$$5 = 8 \cdot 1.25 \cdot \frac{dh}{dt}$$

$$5 = 10 \cdot \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{2}$$

That's it. The swill's level is rising at a rate of $\frac{1}{2}$ *foot per minute* when the swill is 1 foot 3 inches deep. Dig in.

Fasten your seat belt: You're approaching a calculus crossroads

Ready for another common related rates problem? One car leaves an intersection traveling north at 50 *mph*, another is driving west toward the intersection at 40 *mph*. At one point, the north-bound car is three-tenths of a mile north of the intersection and the west-bound car is four-tenths of a mile east of the intersection. At this point, how fast is the distance between the cars changing?

1. **Do the diagram thing. See Figure 12-7.**

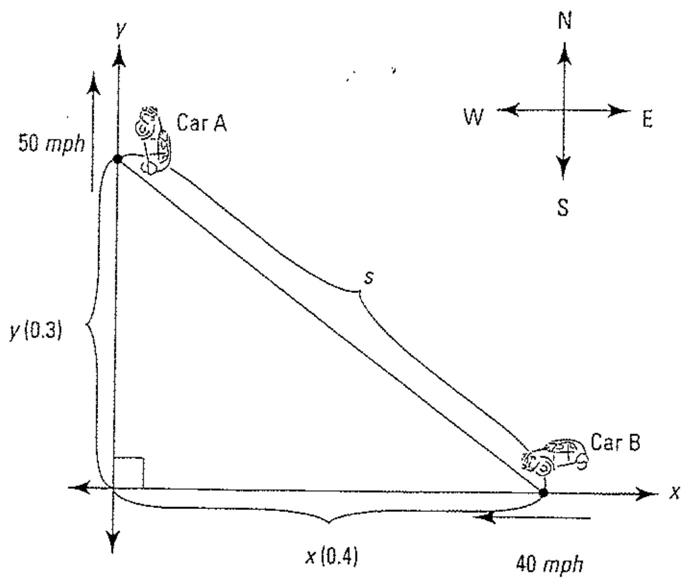

Figure 12-7: Calculus — it's a drive in the country.

TIP

Before going on with this problem, I want to mention a similar problem you may run across if you're using a standard calculus textbook. It involves a ladder leaning against and sliding down a wall. Can you see that the diagram for such a ladder problem would be very similar to Figure 12-7 except that the *y*-axis would represent the wall, the *x*-axis would be the ground, and the diagonal line would be the ladder? These problems are quite similar, but there's an important difference. The distance between the cars is *changing* so the diagonal line in Figure 12-7 is labeled with a variable, *s*. A ladder, on the other hand, has a *fixed* length, so the diagonal line in your diagram for the ladder problem would be labeled with a number, not a variable.

2. **List all given rates and the unknown rate.**

$$\frac{dy}{dt} = 50$$

$$\frac{dx}{dt} = -40$$

$$\frac{ds}{dt} = ?$$

$\frac{dx}{dt}$ is *negative* because car B is going *left*, in the negative *x* direction.

3. **Write the formula that relates the variables in the problem: *x*, *y*, and *s*.**

There's a right triangle in your diagram, so you use the Pythagorean Theorem: $a^2 + b^2 = c^2$. For this problem, *x* and *y* are the legs of the right triangle and *s* is the hypotenuse, so $x^2 + y^2 = s^2$.

TIP

The Pythagorean theorem is used a lot in related rates problems. If there's a right triangle in your problem, it's quite likely that $a^2 + b^2 = c^2$ is the formula you'll need.

Because this formula contains the variables *x*, *y*, and *s*, which all appear in your list of derivatives in Step 2, you don't have to tweak this formula like you did in the trough example.

4. Differentiate with respect to *t*:

$$s^2 = x^2 + y^2$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$ (implicit differentiation with the power rule)

5. Substitute and solve for $\frac{ds}{dt}$.

$x = 0.4$, $y = 0.3$, $\frac{dx}{dt} = -40$, $\frac{dy}{dt} = 50$, and $s = ...$

"Holy devoid distance lacking length, Batman — how can we solve for $\frac{ds}{dt}$ unless we have values for the rest of the unknowns in the equation?"
"Take a chill pill, Robin — just use the Pythagorean Theorem again."

$$s^2 = x^2 + y^2$$
$$s^2 = 0.4^2 + 0.3^2$$
$$= 0.16 + 0.09$$
$$= 0.25$$
$$s = \pm 0.5$$ (square rooting both sides)

You can reject the negative answer because *s* obviously has a positive length. So $s = 0.5$.

Now plug everything into your equation.

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$2 \cdot 0.5 \cdot \frac{ds}{dt} = 2 \cdot 0.4 \cdot (-40) + 2 \cdot 0.3 \cdot 50$$

$$1 \cdot \frac{ds}{dt} = -32 + 30$$

$$\frac{ds}{dt} = -2$$

This negative answer means that the distance, *s*, is decreasing.

Thus, when car A is 3 blocks north of the intersection and car B is 4 blocks east of the intersection, the distance between them is decreasing at a rate of 2 *mph*.