Polynomial Functions and Their Graphs

A polynomial function is an expression that can be written in the form $a_n x^n + a_{n-1}x^{n-1} + \ldots a_2 x^2 + a_1 x + a_0$, where n is a non-negative integer.

Each part of the expression is called a ***term*** and each term has a ***coefficient*** associated with it. The terms are usually written in descending powers of x. The term containing the highest power of x is called the ***leading term*** and its coefficient is referred to as the ***leading coefficient***. The power of x contained in the leading term is called the ***degree*** of the polynomial. Polynomials of the first few degrees have special names as listed in the table below.

Degree	Name	Example of Function
0	constant	$f(x) = 2$
1	linear	$f(x) = x + 2$
2	quadratic	$f(x) = x^2 - 5x + 6$
3	cubic	$f(x) = x^3 - 4x^2 + 2x - 6$
4	quartic	$f(x) = -3x^4 + x$
5	quintic	$f(x) = 5x^5 + 4x^3 - 6$

Polynomial functions with higher degrees are named by their degree. For example, $f(x) = x^8 + 6x^7 + 4x^2$ is called an eighth degree polynomial.

Every polynomial defines a function. Any value for x for which $f(x) = 0$ is the root of the equation and the zero of a function.

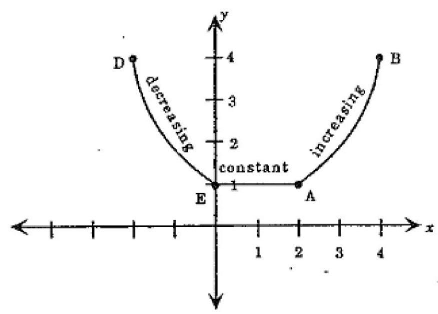
The following features of polynomial functions, f, of degree, n, should be noted.

1. The graph of a polynomial function is continuous. This means the graph has no breaks — you could sketch the graph without lifting your pencil from the paper.

2. The graph of a polynomial function has only smooth turns. The graph of f has at most $(n-1)$ turning points. Turning points are points at which the graph changes from increasing to decreasing or vice versa.

n is degree of poly ∴ has at most one less turning point

A function, f, is increasing in the interval if for any x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.



For the graphs that you investigated, the cubic equation will have at most (3 – 1) turns or two turns. For the graphs that you investigated, the quartic equations will have at most (4 – 1) turns or 3 turns.

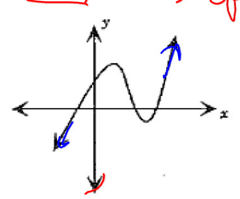
degree of 3

degree 4

3. a) When the degree, n, of a polynomial is ***odd***. → opposite end behaviour (like a linear fcn)

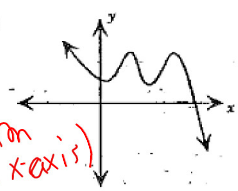
If the leading coefficient is positive (> 0), then the graph falls to the left and rises to the right.

$a > 0$



If the leading coefficient is negative (< 0), then the graph rises to the left and falls to the right.

$a < 0$
(think reflection on x-axis)

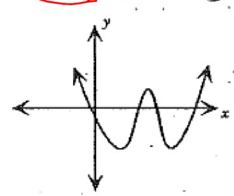


b) When the degree, n, of a polynomial is ***even***. ← same end behaviour

like a parabola

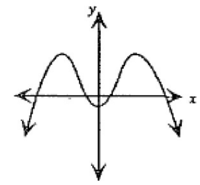
If the leading coefficient is positive (> 0), then the graph rises to the left and right.

$a > 0$



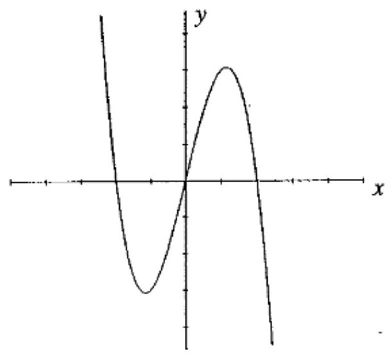
If the leading coefficient is negative (< 0), then the graph falls to the left and right.

$a < 0$
(think reflection)



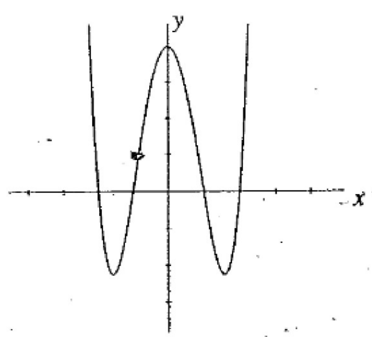
4. The function, f, has at most n real roots. If you have a cubic function you can expect at most three roots.

When you have a quartic function, you can expect it to have at most four roots, and so on.



$f(x) = -x^3 + 4x$
cubic — at most 3 roots

Has to have at least 1

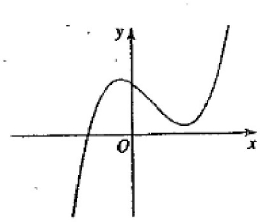


$f(x) = x^4 - 5x^2 + 4$
quartic — at most 4 roots

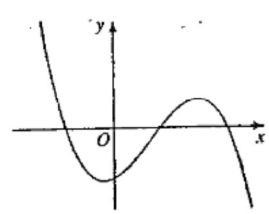
can have none

5. Generally speaking, the graph of a cubic function is shaped like a "sideways S" as shown.

Graph of $f(x) = ax^3 + bx^2 + cx + d$



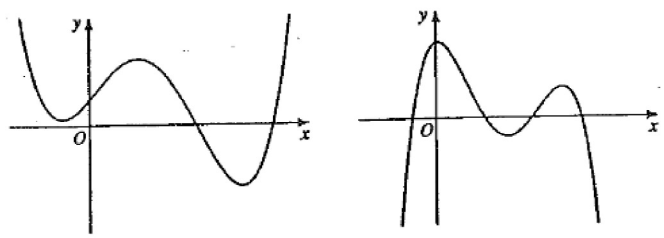
$a > 0$



$a < 0$

Generally speaking, the graph of a quartic equation has a "W shape" or a "M shape."

Graph of $f(x) = ax^4 + bx^3 + cx^2 + dx + e$



$a > 0$ opens up $a < 0$ opens down

6. If a polynomial $f(x)$ has a squared factor such as $(x - c)^2$, then $x = c$ is a ***double root*** of $f(x) = 0$. In this case, the graph of $y = f(x)$ is tangent to the x-axis at $x = c$, as shown in figures (1), (2), and (3) below. multiplicity of 2

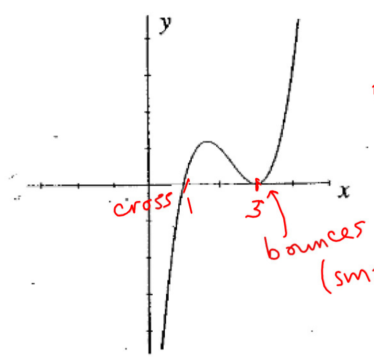


Figure 1
Cubic
$y = (x - 1)(x - 3)^2$

$x = 1$ $x = 3$

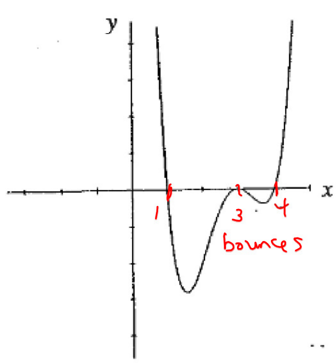


Figure 2
Quartic
$y = (x - 1)(x - 3)^2(x - 4)$

If a polynomial $P(x)$ has a cubed factor such as $(x-c)^3$, then $x = c$ is a ***triple root*** of $P(x) = 0$. In this case, the graph of $y = P(x)$ flattens out around $(c, 0)$ and crosses the x-axis at this point, as shown in figures (4), (5), and (6) below.

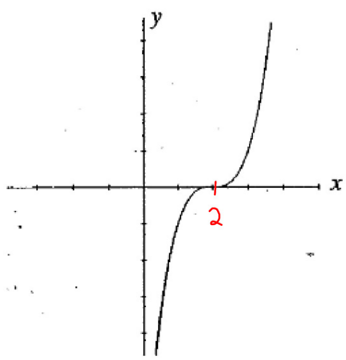


Figure 4
Cubic
$y = (x-2)^3$

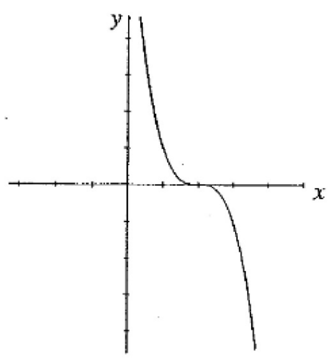


Figure 5
Cubic
$y = -(x-2)^3$

think of vert reflections

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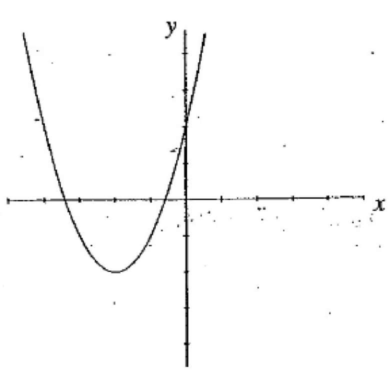
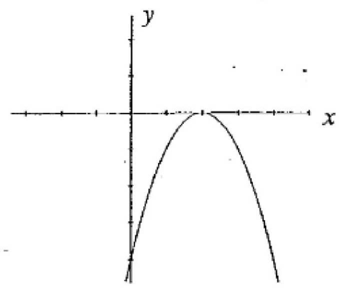
7. When you studied quadratic functions, you were able to determine exactly one maximum or minimum value by determining the coordinates of the vertex. If the graph of the function opened upward, the function had a minimum value. If the graph of the function opened downward, it had a maximum value.



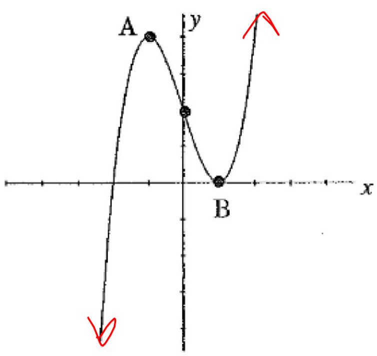
minimum value
$y = (x+2)^2 - 2$



maximum value

$y = -(x + 2)^2$

Cubic functions have neither a maximum or a minimum value but they may have relative maximum or relative minimum values in certain intervals.



$y = x^3 - 3x + 2$

In this figure, you can see that it does not have an absolute maximum y-value (largest y-value) or an absolute minimum y-value (smallest y-value).

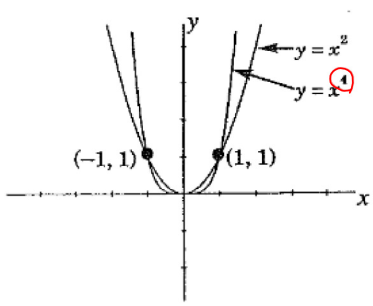
However, it does have a peak at A where a relative maximum value for y is found in a specific interval. In the interval between -2 and 1 the relative maximum is when $y = 4$.

The same graph has a valley at B. This is representing the smallest y-value within a specific interval and is called a relative minimum. Its relative minimum is at $y = 0$ and occurs when $x = 1$.

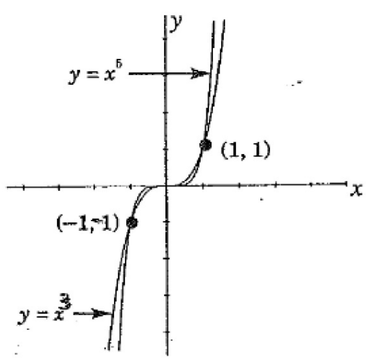
Quartic functions have either an absolute maximum or an absolute minimum value. They can also have relative maximum and/or relative minimum values within intervals.

8. You can use transformations to advantage for some of your sketching of graphs.

The polynomial functions that have the simplest graphs are the monomial functions $f(x) = a_n x^n$. When n is even the graph is similar to the graph of $f(x) = x^2$. When n is odd the graph is similar to the graph of $f(x) = x^3$. Moreover, the greater the value of n, the flatter the graph of a monomial is on the interval $-1 \le x \le 1$.

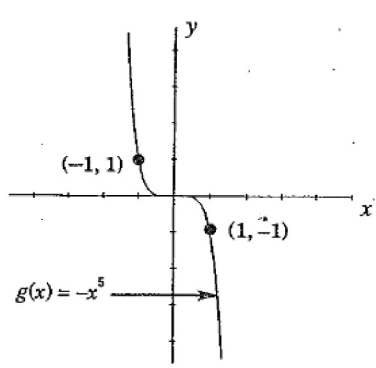


For n even, the graph of $y = x^n$ is tangent to the x-axis at the origin.

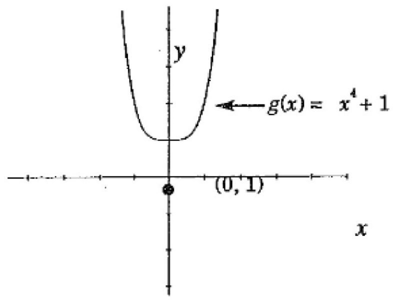


For n odd, the graph of $y = x^n$ crosses the x-axis at the origin.

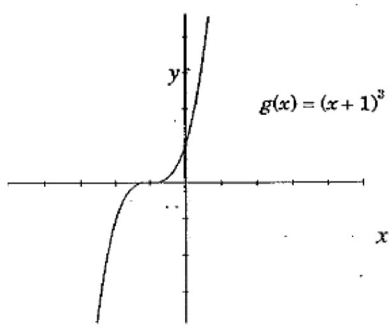
Reflection: To sketch the graph of $g(x) = -x^5$ reflect the graph of $f(x) = x^5$ in the x-axis.



Vertical shift: To sketch the graph of $g(x) = x^4 + 1$ shift the graph of $f(x) = x^4$ up one unit.



Horizontal shift: To sketch the graph of $g(x) = (x + 1)^3$ shift the graph of $f(x) = x^3$ one unit to the left.



The y-intercept of the graph of a function occurs when $x = 0$. The x-intercepts are the ***zeros*** of the function. At an x-intercept, the value of the function is zero.