The maximum volume of a box

A box with no top is to be manufactured from a 30-inch-by-30-inch piece of cardboard by cutting and folding it as shown in Figure 12-1.

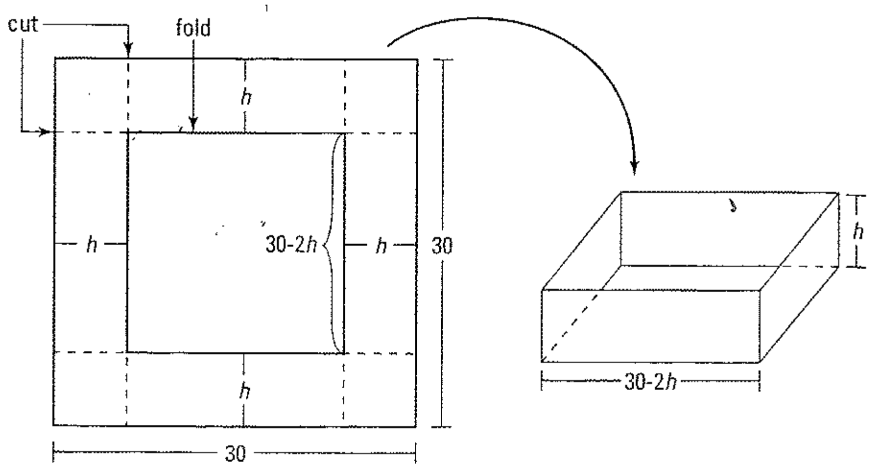

Figure 12-1: The box is made from a 30"-by-30" piece of cardboard by cutting off the corners and folding up the sides.

What dimensions will produce a box with the maximum volume? Mathematics often seems abstract and impractical, but here's an honest-to-goodness practical problem. If a manufacturer can sell bigger boxes for more and is making a hundred thousand boxes, you better believe he or she wants the exact answer to this question. Here's how you do it.

1. **Express the thing you want maximized, the volume, as a function of the unknown, the height of the box (which is the same as the length of the cut).**

   $V = l \cdot w \cdot h$

   $V(h) = (30 - 2h)(30 - 2h) \cdot h$ (You can see in Figure 12-1 that both the *length* and the *width* equal $30 - 2h$.)

   $= (900 - 120h + 4h^2) \cdot h$

   $= 4h^3 - 120h^2 + 900h$

2. **Determine the domain of your function.**

   The height can't be negative or greater than 15 inches (the cardboard is only 30 inches wide, so half of that is the maximum height). Thus, sensible values for h are $0 \leq h \leq 15$. You now want to find the maximum value of $V(h)$ in this interval. You use the method from the "Finding Absolute Extrema on a Closed Interval" section in Chapter 11.

3. **Find the critical numbers of $V(h)$ in the open interval (0, 15) by setting its derivative equal to zero and solving. And don't forget to check for numbers where the derivative is undefined.**

   $V(h) = 4h^3 - 120h^2 + 900h$

   $V'(h) = 12h^2 - 240h + 900$ (power rule)

   $0 = 12h^2 - 240h + 900$

   $0 = h^2 - 20h + 75$ (dividing both sides by 12)

   $0 = (h - 15)(h - 5)$ (ordinary trinomial factoring)

   $h = 15$ or 5

   Because 15 is not in the open interval (0, 15), it doesn't qualify as a critical number (though this is a moot point because you end up testing it below anyway). And because this derivative is defined for all input values, there are no additional critical numbers. So 5 is the only critical number.

4. **Evaluate the function at the critical number, 5, and at the endpoints of the interval, 0 and 15, to locate the function's max.**

$$V(h) = 4h^3 - 120h^2 + 900h$$

$$V(0) = 0$$
$$V(5) = 2000$$
$$V(15) = 0$$

The *extremum* (dig that fancy word for *maximum* or *minimum*) you're looking for doesn't often occur at an endpoint, but it can — so don't fail to evaluate the function at the interval's two endpoints.

So, a height of 5 inches produces the box with maximum volume (2000 cubic inches). Because the length and width equal $30 - 2h$, a height of 5 gives a length and width of $30 - 2 \cdot 5$, or 20, and thus the dimensions of the desired box are 5"-by-20"-by-20". That's it.

The maximum area of a corral — yeehaw!

A rancher can afford 300 feet of fencing to build a corral that's divided into two equal rectangles. See Figure 12-2.

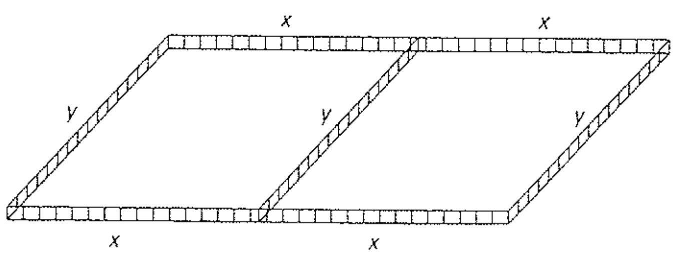

Figure 12-2: Calculus for cowboys — maximizing a corral.

What dimensions will maximize the corral's area? This is another practical problem. The rancher wants to give his animals as much room as possible given the length of fencing he can afford. Like all businesspeople, he wants the most bang for his buck.

1.a. **Express the thing you want maximized, the area, as a function of the two unknowns, *x* and *y*.**

$$A = l \cdot w$$
$$= (2x)(y)$$

In the cardboard box example in the previous section, you can write the volume as a function of *one* variable — which is always what you want. But here, the area is a function of two variables, so Step 1 has two additional sub-steps.

1.b. **Use the given information to relate the two unknowns to each other.**

The fencing is used for seven sections, thus

$$300 = x + x + x + x + y + y + y$$
$$300 = 4x + 3y$$

1.c. Solve this equation for y and plug the result into the y in the equation from Step 1.a. This gives you what you need — a function of one variable.

$$4x + 3y = 300$$
$$3y = 300 - 4x$$
$$y = \frac{300 - 4x}{3}$$
$$y = 100 - \frac{4}{3}x$$
$$A = (2x)(y)$$
$$A(x) = (2x)\left(100 - \frac{4}{3}x\right)$$
$$A(x) = 200x - \frac{8}{3}x^2$$

2. Determine the domain of the function.

You can't have a negative length of fence, so x can't be negative, and the most x can be is 300 divided by 4, or 75. Thus, $0 \le x \le 75$.

3. Find the critical numbers of $A(x)$ in the open interval $(0, 75)$ by setting its derivative equal to zero and solving.

$$A(x) = 200x - \frac{8}{3}x^2$$
$$A'(x) = 200 - \frac{16}{3}x \text{ (power rule)}$$
$$200 - \frac{16}{3}x = 0$$
$$-\frac{16}{3}x = -200$$
$$x = -200 \cdot \left(-\frac{3}{16}\right)$$
$$= \frac{600}{16}$$
$$= 37.5$$

Because A' is defined for all x-values, 37.5 is the only critical number.

4. Evaluate the function at the critical number, 37.5, and at the endpoints of the interval, 0 and 75.

$$A(x) = 200x - \frac{8}{3}x^2$$

$$A(0) = 0$$
$$A(37.5) = 3750$$
$$A(75) = 0$$

Note: Evaluating a function at the endpoints of a closed interval is a standard step in finding an absolute extremum on the interval. However, you could have skipped this step here had you noticed that $A(x)$ is an upside-down parabola and that, therefore, its peak must be higher than either endpoint.

The maximum value in the interval is 3750, and thus, an x-value of 37.5 feet maximizes the corral's area. The length is $2x$, or 75 feet. The width is y, which equals $100 - \frac{4}{3}x$. Plugging in 37.5 gives you $100 - \frac{4}{3}(37.5)$, or 50 feet. So the rancher will build a 75'-by-50' corral with an area of 3750 square feet.